Absence of Asymptotic Freedom in Non-Abelian Models

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The percolation properties of equatorial strips of the two dimensional $O(3)$ nonlinear σ model are investigated numerically. Convincing evidence is found that a sufficently narrow strip does not percolate at arbitrarily low temperatures. Rigorous arguments are used to show that this result implies both the presence of a massless phase at low temperature and lack of asymptotic freedom in the massive continuum limit. A heuristic estimate of the transition temperature is given which is consistent with the numerical data.

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One of the crucial unanswered problems in particle and condensed matter physics concerns the phase diagram of the two dimensional ($2D$) nonlinear σ models [1]. It is widely believed that the nonabelian models with $N \geq 3$ are in a massive phase for any finite inverse temperature β and that this is intimately related to their perturbative asymptotic freedom. Over the years we have brought forth many reasons why we think that these beliefs are unfounded [2–4], but the absence of a mathematical proof combined with ambiguous numerical results left the issues wide open. In the present letter we would like to provide convincing numerical evidence that in fact the $2D$ $O(3)$ model possesses a massless phase for sufficiently large β and give a rigorous proof that this is incompatible with asymptotic freedom in the massive phase. We will also give a heuristic explanation of why and where the phase transition happens.

The models we are considering consist of unit length spins s taking values on the sphere S^{N-1}, placed at the sites of a $2D$ regular lattice. These spins interact ferromagnetically with their nearest neighbors. Let $\langle ij\rangle$ denote a pair of neighboring sites. We will consider two types of interactions between neighbouring spins:

- Standard action (s.a.): $H_{ij} = -s(i)\cdot s(j)$
- Constrained action (c.a.): $H_{ij} = -s(i)\cdot s(j)$ for $s(i)\cdot s(j) \geq c$ and $H_{ij} = \infty$ for $s(i)\cdot s(j) < c$ for some $c \in [-1, 1)$.

Almost a decade ago we showed [5] that one can rephrase the issue regarding the existence of a soft phase in these models as a percolation problem and in fact this is the reason we introduced the c.a. model (please note that the c.a. model has the same perturbative expansion as the s.a. model and possesses instantons). Namely let $\epsilon = \sqrt{2(1-c)}$ and S_ϵ the set of sites such that $|s{\cdot}n| < \epsilon/2$ for some given unit vector n. Our rigorous result was that if on the triangular (T) lattice the set S_ϵ does not contain a percolating cluster, then the $O(N)$ model must be massless at that c. For the abelian $O(2)$ model we could prove the absence of percolation of this equatorial set S_ϵ for c sufficiently large [5] (modulo certain technical assumptions which were later eliminated by M. Aizenman [6]). For the nonabelian cases we could not give a rigorous proof. We did however present certain arguments [7,8] explaining why the percolating scenario seemed unlikely.

In this letter we will give convincing numerical evidence that a sufficiently narrow equatorial strip does not percolate for any c. We will also show that via a rigorous inequality derived by us in the past [9], the existence of a finite β_{crt} in the s.a. model on the square (S) lattice is incompatible with the presence of asymptotic freedom in the massive continuum limit.

For clarity we will review first a few crucial points regarding percolation in $2D$:

a) Let A be a subset of the lattice defined by the spin lying in some subset $\mathcal{A} \subset S^{N-1}$ and $\tilde{A}$ its complement. Then with probability 1 A and $\tilde{A}$ do not percolate at the same time. (For this point it is crucial that the lattice is self matching, hence our use of the T instead of the S lattice; on the latter an ordinarily connected cluster can be stopped by a cluster connected only $\star$-wise, i.e. also diagonally.) This fact has been proven rigorously only for special cases like the $+$ and $-$ clusters of the Ising model, but is believed to hold generally.

b) If neither A nor its complement $\tilde{A}$ percolate, then the expected size of the cluster of A attached to the origin, denoted by $\langle A\rangle$ of A, diverges; the same holds for its complement $\tilde{A}$ (Russo's lemma [10]). (If $\tilde{A}$ percolates, then $\langle A\rangle$ is finite.)

Let then $\mathcal{P}_\epsilon$ (the union of the polar caps) be the complement of $\mathcal{S}_\epsilon$ (the equatorial strip of width ϵ). According to the discussion above, either S_ϵ percolates, or P_ϵ percolates, or neither S_ϵ nor P_ϵ percolates and then both have divergent mean size (we shall call this third possibility in short *formation of rings*). Consider a sequence of tori of

increasing size L and the mean cluster size of the set A corresponding to a subset $\mathcal{A} \subset S^{N-1}$ of positive measure. If A percolates $\langle A \rangle = O(L^2)$, if its complement percolates $\langle A \rangle$ will approach a finite nonzero value, and if A forms rings $\langle A \rangle = O(L^{2-\eta})$ for some $\eta > 0$. Therefore, if we define the ratio

$$r = \langle P_\epsilon \rangle / \langle S_\epsilon \rangle, \tag{1}$$

for $L \to \infty$ it should either go to 0, or to ∞ or to some finite nonzero value depending on which one of the three possibilities is realized; the latter possibility assumes that η is the same for P_ϵ and S_ϵ, as indicated by our numerics and consistent with scaling.

In Fig.1 we show the numerical value of the ratio r as function of c for $\beta = 0$ for four values of ϵ for the c.a. model on a T lattice.

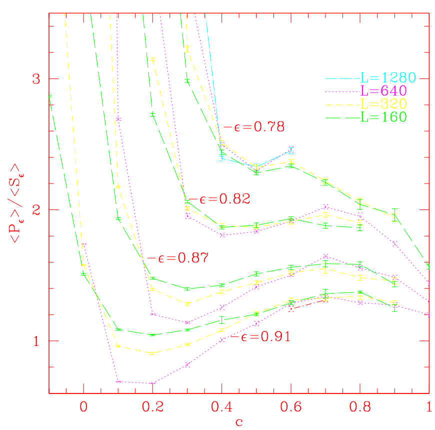

FIG. 1. Ratio $\langle P_\epsilon \rangle / \langle S_\epsilon \rangle$ for various ϵ values versus c

The results were obtained from a Monte Carlo (MC) investigation using an $O(3)$ version of the Swendsen-Wang cluster algorithm and consist of a minimum of 20,000 lattice configurations used for taking measurements. For each value of ϵ we studied $L = 160$, 320 and 640 (for $\epsilon = .78$ we also studied $L = 1280$). Three distinct regimes are manifest for each of the four values of ϵ investigated:

- For small c, r is increasing with L, presumably diverging to ∞ (region 1).
- For intermediate c, r is decreasing with L, presumably converging to 0 (region 2).
- For c sufficiently large depending upon ϵ, r becomes independent of L, just as it does at the crossings from region 1 into 2.

Consequently for these values of ϵ for small c P_ϵ percolates, for intermediate c S_ϵ percolates and for sufficiently large c both P_ϵ and S_ϵ form rings. In Fig.2 we present the phase diagram of the c.a. model on the T lattice for $\beta = 0$. The dashed line D represents the minimal equatorial width above which S_ϵ percolates. For $c = -1$ (no constraint) the model reduces to independent site percolation, for which the percolation threshold is known rigorously to be $\epsilon = 1$. The rest of the diagram represents qualitatively the results of our investigation of the ratio r, such as those shown in Fig.1. Two features of this diagram are worth emphasizing:

1. An equatorial strip of width less than approximately $\epsilon = .76$ *never* percolates.
2. For approximately $c > 0.4$ a new phase opens up in which both P_ϵ and S_ϵ form rings (the dotted line separates it from percolation of P_ϵ).

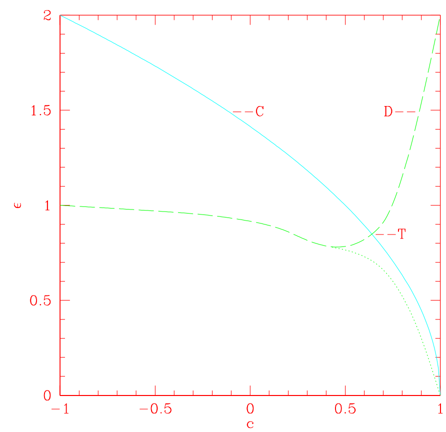

FIG. 2. Phase diagram of the $O(3)$ model on the T lattice

For clarity let us briefly review the argument indicating that this phase diagram is incompatible with the existence of a massive phase for arbitrarily large c. Choosing an arbitrary unit vector n, one introduces Ising variables $\sigma = \pm 1$. The s.a. Hamiltonian becomes:

$$H_{ij} = \sigma_i \sigma_j |s_\parallel(i) s_\parallel(j)| + s_\perp(i) \cdot s_\perp(j) \tag{2}$$

where $s_\parallel(i) = s(i) \cdot n$ and $s_\perp(i) = (s(i) \times n) \times n$. One thus obtains an induced Ising model for which the Fortuin-Kastleyn (FK) representation [11] is applicable. In this representation the Ising system is mapped into a bond percolation problem: between any like neighboring Ising spins a bond is placed with probability $p = 1 - \exp(-2\beta s_\parallel(i) s_\parallel(j))$. For the c.a. model a bond is also placed if after flipping one of the two neighboring Ising spins the constraint $s(i) \cdot s(j) \geq c$ is violated. The FK representation relates the mean cluster size of the site clusters joined by occupied bonds (FK-clusters) to the Ising magnetic susceptibility. In a massive phase the latter must remain finite. Hence, if the FK-clusters have divergent mean size, the original $O(3)$ ferromagnet must be massless (the Ising variables σ are local functions of the originally spin variables s).

Now notice that by construction for the c.a. model the FK-clusters with say $\sigma = +1$ must contain all sites with $s(i) \cdot n > \sqrt{(1-c)/2}$. Therefore the model must be massless if clusters obeying this condition have divergent mean size. But the polar set P_ϵ consists of two disjoint

components P_ϵ^+ (north) and P_ϵ^- (south). For $c > (1-\epsilon^2)/2$ there are no clusters containing both elements of P_ϵ^+ and P_ϵ^-. Hence if for such values of c clusters of P_ϵ form rings, so do clusters of P_ϵ^+ separately and hence the $O(3)$ model must be massless. The curve C given by $c = (1-\epsilon^2)/2$ is the solid line in Fig.2. The point T at the intersection of the curves D and C gives an upper bound for c_{crt}, the value of c above which the c.a. model must be massless.

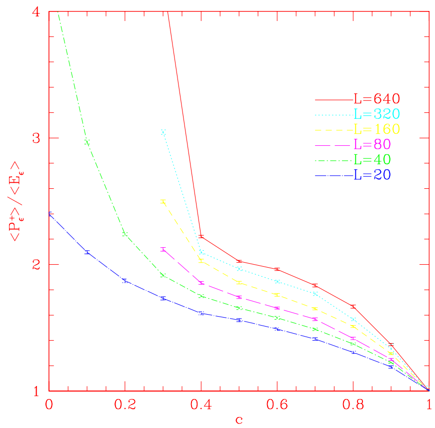

FIG. 3. The ratio of the mean cluster size of a polar cap of height .75 to that of an equatorial strip of the same height

To verify the phase diagram in Fig.2 we also measured (at $\beta = 0$) the ratio of the mean cluster size of the set $P_{\epsilon'}^+$ with $\epsilon' = .5$ to that of the set S_ϵ with $\epsilon = 0.75$, for which the equatorial strip never percolates (Fig.3) (ϵ and ϵ' ar chosen such that the two sets have equal density). The data indicate that for some intermediate values of c $P_{\epsilon'}^+$ forms rings while S_ϵ has finite mean size; this region terminates around $c = 0.4$, where also S_ϵ starts forming rings. The larger average cluster size of the polar cap compared to the strip of the same area is probably due to the fact that the strip has a larger boundary than the polar cap. This is in agreement with a general conjecture stated in [7], namely that for c sufficiently large, if two sets have equal area but different perimeters, the one with the smaller perimeter will have larger average cluster size. For the case at hand, this is apparently true for all values of c.

The general belief, which we criticized in ref. [3],is that there is a fundamental difference between abelian and nonabelian models. To test this belief we studied the ratio r for the c.a. $O(2)$ model on the T lattice. The phase diagram is shown in Fig.4. Since in the $O(2)$ model the set $\mathcal{P}_\epsilon$ can also be regarded as a set $\mathcal{S}_{\tilde\epsilon}$ where $\tilde\epsilon = \sqrt{(4-\epsilon^2)}$, certain features of that diagram follow from rigorous arguments. For instance it is clear that in the c.a. model there exist two curves C and $\tilde C$ and in the region to their right the model must be massless [5,6]. The precise location of the curves D (or $\tilde D$) must be determined numerically, something which we did not do. We did verify though that the ring formation region begins around $c = -0.5$.

In our opinion the arguments and numerical evidence provided so far give strong indications that the c.a. models on the T lattice possess a massless phase. Universality would suggest that a similar situation must exist for the s.a. models on the T and S lattices. To test universality we measured on the S lattice the renormalized coupling both on thermodynamic lattices in the massive phase and in finite volume in the presumed critical regime (as in [12]). Our data for the c.a. model on the S lattice only determine an interval (about .5 to .7) in which the massless phase of the model sets in; we tried to see if we could get a similar L dependence for the renormalized coupling in the s.a. model at a suitable β as for $c = .61$ in the c.a. model at $\beta = 0$. This seems to be indeed the case for β roughly 3.4. We went only up to $L = 640$, hence this equivalence between c and β should be considered only as a rough approximation, but there seems to be no doubt that the two models have the same continuum limit.

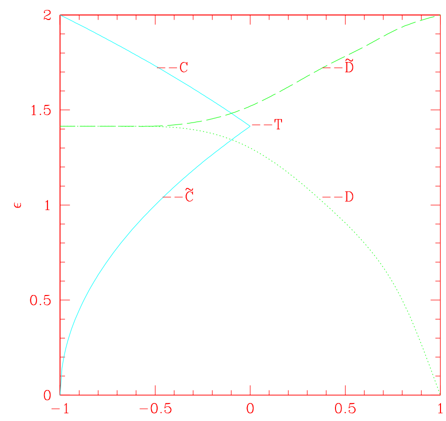

FIG. 4. The phase diagram for the $O(2)$ model on the T lattice

It is intersting to note that there is a heuristic explanation for both the existence of a massless phase in the s.a. $O(3)$ model and for the value of β_{crt}. Indeed it is known rigorously that in $2D$ a continuous symmetry cannot be broken at any finite β. In a previuos paper [4] we showed that the dominant configurations at large β are not instantons but superinstantons (s.i.). In principle both instantons and s.i. could enforce the $O(3)$ symmetry. In a box of diameter R the former have a minimal energy $E_{inst} = 4\pi$ [13] while the latter $E_{s.i.} = \delta^2\pi/\ln R$, where δ is the angle by which the spin has rotated over the distance R. Now suppose that β_{crt} is sufficiently large for classical configurations to be dominant. Then let us choose $\delta = 2\pi$ (restoration of symmetry) and ask how large must R be so that the superinstanton configuration has the same energy as one instanton. One finds $\ln R = \pi^2$. But in the Gaussian approximation

$$\langle s(0)\cdot s(x)\rangle \approx 1 - \frac{1}{\beta\pi}\ln x \tag{3}$$

Thus restoration of symmetry occurs for $\ln x \approx \pi\beta$. This simpleminded argument suggests that for $\beta \geq \pi$ instantons become less important than s.i.. Now in a gas of s.i. the image of any small patch of the sphere forms rings, hence the system is massless. While this is not a quantitative argument, we believe it captures qualitatively what happens: a transition from localized defects (instantons) to s.i..

Next let us discuss the connection between a finite β_{crt} and the absence of asymptotic freedom. It follows from our earlier work concerning the conformal properties of the critical $O(2)$ model [9]. We refer the reader for details to that paper and give only an outline of the argument. The s.a. lattice $O(N)$ model possesses a conserved isospin current. This currrent can be decomposed into a transverse and longitudinal part. Let $F^T(p)$ and $F^L(p)$ denote the thermodynamic values of the 2-point functions of the transverse and longitudinal parts at momentum p, respectively. Using reflection positivity and a Ward identity we proved that in the massive continuum limit the following inequalities must hold for any $p \neq 0$:

$$0 \leq F^T(p) \leq F^T(0) = F^L(0) \leq F^L(p) = 2\beta E/N \quad (4)$$

Here E is the expectation value of the energy

$$E = \langle s(i) \cdot s(j) \rangle$$

at inverse temperature β. Since $E \leq 1$ it follws that if $\beta_{crt} < \infty$ $F^T(0) - F^T(p)$ cannot diverge for $p \to \infty$ as required by perturbative asymptotic freedom. In fact, for $\beta_{crt} = 3.4$ (which is a plausible guess) $F^T(p)$ must be less than 2.27, in violation of the form factor computation giving $F^T(0) - F^T(\infty) > 3.651$ [14].

Since the implications of our result, that for the c.a. model a suffciently narrow equatorial strip never percolates, are so dramatic, the reader may wonder how credible are the numerics. The usual suspect, the random number generator, should not be important since precision is not the issue here. The only debatable point is whether our results represent the true thermodynamic behaviour for $L \to \infty$ or are merely small volume artefacts. While we cannot rule out rigorously that scenario, certain features of the data make it highly implausible:

- Small volume effects should set in gradually, while the data in Fig.1 indicate a rather abrupt change from a region where r is decreasing with L to one where r is essentially independent of L.

- For $c \to 1$ at fixed L, r must approach the 'geometric' value $r = 2/\epsilon - 1$. As can be seen, in all the cases studied, throughout the 'ring' region r is clearly larger than this value, while it should go to 0 if S_ϵ percolated.

- In Fig.3 there is no indication of the ratio going to 0 for increasing L. Moreover the dramatic change in slope around $c = .4$ indicates that the polar cap $P_{\epsilon'}$ starts forming rings at a smaller value of c than the equatorial strip S_ϵ.

Thus we doubt very much that the effects we are seeing represent small volume artefacts. Moreover, if s_z remained massive at low temperature and in fact an arbitrarily narrow equatorial strip percolated, one would have to explain away our old paradox [7,8]: if such a narrow strip percolated, an even larger strip would percolate and on it one would have an induced $O(2)$ model in its massless phase, in contradiction to the Mermin-Wagner theorem.

Consequently we are confident that the phase diagram in Fig.2 represents the truth, that a soft phase exists both in the s.a. and the c.a. model and that the massive continuum limit of the $O(3)$ model is not asymptotically free. In a previous paper [4] we have already shown that in nonabelian models even at short distances perturbation theory produces ambiguous answers. The present result sharpens that result by eliminating the possibility of asymptotic freedom.

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